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On the transport equations for a one-component relativistic gas

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Abstract. We use a generalization of the relativistic Grad equations when time reversal symmetry for the scattering matrix does not hold, in order to obtain a new set of transport equations for a gas in the vicinity of the Jüttner–Synge equilibrium. In the particular 'Fourier situation' our heat equation very much resembles that of Cattaneo and Vernotte. However, when PT invariance is introduced, this equation reduces to the usual Eckart-like one. Nearly identical conclusions are obtained for a 'Bel-like' gas.

1. Introduction

In two fundamental papers (Anderson 1970, Stewart 1971) a relativistic version of the Grad method of moments for constructing approximate solutions of the Boltzmann equation, has been introduced. As has been remarked by Anderson, this method has some advantages over the older Chapman–Enskog–Hilbert method, and, in particular, it leads to the phenomenological 'Eckart' transport equations (Eckart 1940) for heat propagation and the shear and bulk viscosities†

$$q^{\rho} = -\kappa h^{\rho\mu}(\partial_{\mu}T + T\dot{u}_{\mu}) \tag{1}$$

$$\pi^{\alpha\mu} = -\eta\sigma^{\alpha\mu}. \tag{2}$$

In Anderson (1970) and Stewart (1971), and also in earlier work by Chernikov (1964) and Marle (1969), the assumption was made that the 'scattering matrix' is invariant under both time reversal and space reflection; that is, if $W(p, q; p', q')f(x, p)f(x, q)\omega_p\omega_q\omega_{p'}\omega_{q'}\eta$ gives the number of collisions in the four-volume η at x^{ρ}, between particles with initial momenta p, q in the ranges ω_p, ω_q and final momenta p', q' in the ranges $\omega_{p'}$, $\omega_{q'}$, the assumption made (which we shall call 'PT invariance') is that

$$W(p, q; p', q') = W(p', q'; p, q) \tag{3a}$$

the four-momentum being conserved in collisions, i.e.,

$$p^{\alpha} + q^{\alpha} = p'^{\alpha} + q'^{\alpha}. \tag{3b}$$

† We shall use signature +2 for the metric of the space–time manifold V_4, and the Einstein summation convention, with $\alpha, \beta, \ldots = 0, 1, 2, 3$ and $i, j \ldots = 1, 2, 3$. $h^{\rho\mu}$ is the projection tensor $h^{\rho\mu} \equiv g^{\rho\mu} + u^{\rho}u^{\mu}$, and $\dot{u}^{\mu}$ is the four-acceleration of the fluid, $\dot{u}^{\mu} \equiv u^{\rho}\nabla_{\rho}u^{\mu}$, u^{ρ} being the field of fluid mean four-velocities.

The aim of this paper is to explore the consequences of relaxing these assumptions. We believe that this paper has an interest which is not only formal, because as Bel and Martín (1975) have shown, in the framework of a predictive relativistic mechanics, hypothesis (3*b*) (conservation of 'kinetic four-momentum' in collisions) appears to be justified only when the system is conservative. (Up to the first order in perturbation theory, an example of a conservative system would be a short-range Poincaré-invariant predictive system, which satisfies Newton's law and is invariant under time reversal.) In particular, if, when studying the electromagnetic interaction from this point of view, one adopts the retarded Lienard–Wiechert potentials, the system is not T-invariant and the theory does not lead generally to (3*b*).

Of course, the scattering experiments performed put a severe upper bound on this non-conservation over the usual range of energies; say, the quantities

$$t^{\mu} \equiv p^{\mu} + q^{\mu} - p'^{\mu} - q'^{\mu}$$

must be very 'weak'. This fact will be used in the sequel for generalizing the usual 'normal solution' approximation. Physically, it is possible that this kind of 'non-T-invariant' interaction plays an important role in 'initial' ($R/R_0 \ll 1$) cosmological situations, but in fact it is not clear that a classical (non-quantal) theory, such as the one developed in this paper, could be used for this purpose.

The kind of physical situation that we had in mind is then, an idealized 'cosmological collisional gas' which is not very far from equilibrium. It is well known that the Jüttner–Synge (Boltzmann equilibrium) function is not compatible with Robertson–Walker space–times, and so in § 5 we carry out the calculation for a gas very close to the Bel ('Liouville equilibrium') situation, which elsewhere (Alvarez 1976) has proved to be compatible with expanding universes.

The first problem that we must solve is, then, that of obtaining information concerning non-equilibrium solutions of the relativistic Boltzmann equation, when the distribution functions differ from the local equilibrium distribution by a small amount, in the general situation where no symmetries are imposed on the scattering matrix. We shall, then, in § 2, generalize the Grad method of Anderson and Stewart, introducing in § 3 two types of approximation, which we think are simpler generalizations of the usual 'normal solution approximation'. Then we obtain the transport equations corresponding to these approximations, studying their limits when restoring the symmetries (3) of the scattering matrix.

Then, in § 5, we shall study the same problem, but taking the local Bel function (Bel 1969, Alvarez 1976) as the zero-order distribution function, instead of the Jüttner–Synge function, in view of possible applications in cosmology.

2. The generalized Grad equations

Our purpose is to solve the Boltzmann equation

$$\mathscr{L}(\boldsymbol{X})f(x, p) = Q(f, f) \tag{4}$$

where $\mathscr{L}(\boldsymbol{X})$ is the Lie derivative with respect to $\boldsymbol{X}$, $\boldsymbol{X}$ being a vector on the phase space $P(V_4)$ with components

$$\boldsymbol{X} \equiv (p^{\alpha}, Q^{j} \equiv -P^{j}_{\mu\nu} p^{\mu} p^{\nu}).$$

The one-particle phase space $P(V_4)$ is defined as

$$P(V_4) \equiv \{(x, \boldsymbol{p}), x \in V_4, \boldsymbol{p} \in P_x\},$$

P_x being the mass hyperboloid at the point x of the space–time manifold V_4, i.e. the set of vectors,

$$P_x \equiv \{\boldsymbol{p} \in T_x, \boldsymbol{p}^2 = -m^2, p^0 > 0\}.$$

The second member of the Boltzmann equation is written as (Lichnerowicz and Marrot 1940)

$$Q(f, f) \equiv \int \{f(x, p')f(x, q')W_{\mathrm{I}} - f(x, p)f(x, q)W_{\mathrm{D}}\}\omega_{p'}\omega_{q'}\omega_q \tag{5}$$

ω_p being the invariant volume element on P_x,

$$\omega_p \equiv \sqrt{|g|}\frac{\mathrm{d}p^1 \wedge \mathrm{d}p^2 \wedge \mathrm{d}p^3}{|g_{0\alpha}p^\alpha|}$$

and W_{I} and W_{D} being related to the scattering matrix W, defined in the introduction, by:

$$W_{\mathrm{D}} \equiv W(p, q; p', q') \qquad W_{\mathrm{I}} \equiv W(p', q'; p, q).$$

Obviously, when PT invariance (3) holds, one must write $W_{\mathrm{D}} = W_{\mathrm{I}}$.

In order to solve the Boltzmann equation (4), we shall consider only small deviations from 'equilibrium', $f_0(x, p)$, which we consider, in this paper, to be a local Jüttner–Synge function†, i.e.,

$$f_0(x, p) = B(x)\,\mathrm{e}^{\lambda_\rho p^\rho}, \tag{6}$$

where $B(x)$ is related to the 'particle number density' $n(x^\rho)$ by

$$B(x) = \frac{n\gamma}{4\pi m^3 K_2(\gamma)},$$

γ being defined through $\lambda^2 \equiv \boldsymbol{\lambda}^2 \equiv m^{-2}\gamma^2$; that is to say, we expand the distribution function of the gas, $f(x, p)$ as

$$f = f_0 \sum_{n=0}^{\infty} a_{\boldsymbol{\alpha}_n}(x)H^{\boldsymbol{\alpha}_n}(x, p), \tag{7}$$

$H^{\boldsymbol{\alpha}_n}(x, p)$ being an orthogonal set of polynomials in the momentum, with f_0 as the weight function, n representing the order of the polynomial and $\boldsymbol{\alpha}_n$ a generic 'vector' index, i.e.,

$$H^{\boldsymbol{\alpha}_n} \equiv H^{\alpha_1 \cdots \alpha_n} \qquad a_{\boldsymbol{\alpha}_n} \equiv a_{\alpha_1 \ldots \alpha_n}.$$

We impose the orthogonality of these polynomials:

$$\langle H^{\boldsymbol{\alpha}_n}, H^{\boldsymbol{\alpha}_r}\rangle \equiv \int f_0(x, p)H^{\boldsymbol{\alpha}_n}H^{\boldsymbol{\alpha}_r}\omega_p = M^{\boldsymbol{\alpha}\boldsymbol{\beta}}\delta^{nr} \tag{8}$$

and construct the polynomials by a Schmidt orthogonalization procedure. We begin with $H^0 = 1$, and take (details are given in Stewart 1971)

$$H^\mu = p^\mu - \alpha_1^\mu(x^\rho).$$

† We shall assume that all the particles of the relativistic gas possess the same mass, say $m > 0$.

Then, as (8) implies that $\langle H^0, H^\mu\rangle = 0$, we get

$$\alpha_1^\mu = N_0^\mu A_0^{-1}$$
$$M^{\alpha\beta} \equiv \langle H^\alpha, H^\beta\rangle = T_0^{\alpha\beta} - N_0^\alpha N_0^\beta A_0^{-1}, \tag{9}$$

where A_0, N_0^μ and $T_0^{\alpha\beta}$ are the first three moments of $f_0(x, p)$:

$$A_0 \equiv \int f_0(x,p)\omega_p \qquad N_0^\mu \equiv \int f_0(x,p)p^\mu\omega_p \qquad T_0^{\alpha\beta} \equiv \int f_0(x,p)p^\alpha p^\beta \omega_p.$$

One then decomposes

$$H^{\alpha\beta} = p^\alpha p^\beta - \alpha_{1\gamma}^{\alpha\beta} H^\gamma - \beta^{\alpha\beta}$$

and gets equations similar to (9) for $\alpha_1^{\alpha\beta}$, $\beta^{\alpha\beta}$ and $M^{\alpha\beta\gamma\delta}$ by imposing $\langle H^{\alpha\beta}, H^0\rangle = \langle H^{\alpha\beta}, H^\delta\rangle = 0$. Finally, we also need

$$H^{\alpha\beta\gamma} = p^\alpha p^\beta p^\gamma - \alpha_{1\ \delta\epsilon}^{\alpha\beta\gamma} H^{\delta\epsilon} - \beta^{\alpha\beta\gamma}_{\ \ \ \delta} H^\delta - \gamma^{\alpha\beta\gamma}. \tag{9b}$$

If we multiply (7) by H^{α_n}, and integrate over the mass shell, $P_x(p)$, we get

$$M_n^{\alpha\beta}(x^\rho)a_{\beta_n}(x) = \int f(x,p)H^{\alpha_n}(x,p)\omega_p. \tag{10}$$

The first three equations of the set (10) (i.e. those corresponding to $n = 0, 1, 2$) are

$$A_0 a_0 = A$$
$$M^{\alpha\beta}a_\beta = N^\alpha - N_0^\alpha A A_0^{-1} \tag{11}$$
$$M^{\alpha\beta\gamma\delta}a_{\gamma\delta} = T^{\alpha\beta} - \alpha_{1\ \gamma}^{\alpha\beta}(N^\gamma - N_0^\gamma A A_0^{-1}) - T_0^{\alpha\beta} A A_0^{-1}$$

where A, N^α and $T^{\alpha\beta}$ represent the first three moments of the distribution function $f(x, p)$.

We further assume that some of these moments are related to the moments of $f_0(x, p)$ by simple relations; explicitly, we impose the 'matching conditions'

$$\delta A \equiv A(x^\rho) - A_0(x^\rho) = 0$$
$$u_\rho \delta N^\rho \equiv u_\rho(N^\rho - N_0^\rho) = 0 \tag{12}$$

u^ρ being the unitary ($\boldsymbol{u}^2 = -1$) vector colinear to N_0^ρ (i.e. to λ^ρ in (6)); that is, the mean four-velocity of the fluid. The physical meaning of the matching conditions (12) is that the natural requirements that the particle number density and the trace of the energy–momentum tensor ($T_\alpha^\alpha = -m^2 A(x)$) using f, must be the same functions of their arguments as they are when calculated using f_0. In the usual ('T-invariant') situation (Anderson 1970) one needs three more conditions to completely specify the problem; conditions that define the local rest frame, i.e. fix u^ρ, and are normally taken to be

$$N^\alpha = N_0^\alpha. \tag{12b}$$

But as we shall see later, these 'Eckart' matching conditions are not compatible with our generalized form of Grad equations, so we will choose the weaker form (12). We decompose, as usual, the different tensorial quantities which appear in the calculation in terms of the two tensors at our disposal, i.e. $g_{\alpha\beta}$ and u_μ:

$$\begin{aligned}
a_\mu &\equiv K_1 \,.\, u_\mu + K_{2\mu} \qquad (K_{2\mu}u^\mu = 0)\\
a_{\mu\nu} &\equiv L_0 \,.\, u_\mu u_\nu + L_1 g_{\mu\nu} + 2L_{2(\mu}u_{\nu)} + L_{3\mu\nu} \qquad (L_{2\mu}u^\mu = L_{3\mu}{}^\mu = L_{3\mu\nu}u^\nu = 0)\\
\alpha_1^{\alpha\beta}{}_\gamma &\equiv c_1 u^\alpha u^\beta u_\gamma + c_2 g^{\alpha\beta}u_\gamma + c_3 u^{(\alpha}g^{\beta)}{}_\gamma\\
M^{\alpha\beta} &\equiv M_{11}u^\alpha u^\beta + M_{12}g^{\alpha\beta}\\
M^{\alpha\beta\gamma\delta} &\equiv M_1 u^\alpha u^\beta u^\gamma u^\delta + M_2 g^{\alpha\beta}u^\gamma u^\delta + M_3 u^\alpha u^\beta g^{\gamma\delta}\\
&\quad + M_4 u^{(\alpha}g^{\beta)(\delta}u^{\gamma)} + M_5 g^{\alpha(\gamma}g^{\delta)\beta} + M_6 g^{\alpha\beta}g^{\gamma\delta}
\end{aligned} \tag{13}$$

and also the moments of the two distribution functions (the zeroth-order one, $f_0(x, p)$, and the one we wish to calculate, $f(x, p)$:

$$\begin{aligned}
N_0^\alpha &\equiv n_0 u^\alpha \equiv \frac{\rho_0}{m}u^\alpha\\
T_0^{\alpha\beta} &\equiv (\mu_0 + p_0)u^\alpha u^\beta + p_0 g^{\alpha\beta}\\
N^\alpha &\equiv n u^\alpha + j^\alpha \qquad (j^\alpha u_\alpha = 0)\\
T^{\alpha\beta} &\equiv (\mu + p)u^\alpha u^\beta + p g^{\alpha\beta} + 2q^{(\alpha}u^{\beta)} + \pi^{\alpha\beta} \qquad (q^\alpha u_\alpha = \pi^\alpha{}_\alpha = \pi^{\alpha\beta}u_\beta = 0).
\end{aligned} \tag{14}$$

Using the matching conditions (12), we first obtain

$$n = n_0 \qquad \mu - 3p = \mu_0 - 3p_0,$$

and by substituting the decompositions (13) and (14) in (11), and after some algebra, one gets:

$$\begin{aligned}
&a_0 = 1\\
&K_1 = 0\\
&M_{12}K_2^\alpha = j^\alpha \dagger\\
&\pi^{\alpha\beta} = M_5 L_3^{\alpha\beta}\\
&2q^\alpha = (2M_5 - M_4)L_2^\alpha + c_3 M_2 K_2^\alpha\\
&\tau \equiv p - p_0 = \tfrac{1}{4}L_0(3M_2 + M_5).
\end{aligned} \tag{15}$$

The set of equations (15) allows us, then, to relate the basic 'transport' quantities, i.e. the 'diffusion flux' j^α, the 'heat flux' q^α and the 'bulk viscosity' τ, to the orthogonal polynomials defined by (8) and the coefficients a_{α_n} in the expansion (7) up to $n = 2$. Up to now we have not used the Boltzmann equation. The basic idea of the Grad method of moments is to truncate the series (7) at $n = 3$ (i.e. to suppose $a_{\alpha_n} = 0$ if $n \geqslant 3$), to substitute it in the Boltzmann equation (4), (5), and linearize it in order to obtain equations for the first two coefficients a_{α_n} ($n = 1, 2$). By substitution in (15) one then gets the linear transport equations.

† It is to be noted that when one imposes the usual 'Eckart' matching condition, $\delta N^\rho = 0$, i.e. $j^\alpha = 0$, one gets $K_2^\alpha = 0$, that is,

$$a_\mu = 0. \tag{15b}$$

To accomplish this operation, one starts from the Boltzmann equation, multiplies it by $p^{\alpha_n} \equiv p^{\alpha_1} \ldots p^{\alpha_n}$ and integrates, to obtain

$$\int p^{\alpha_n} \mathscr{L}(\boldsymbol{X}) f(x, p) \omega_p = \int p^{\alpha_n} (W_{\mathrm{I}} f(p') f(q') - W_{\mathrm{D}} f(p) f(q)) \omega^4 \tag{16}$$

with

$$\omega^4 \equiv \omega_p \wedge \omega_q \wedge \omega_{p'} \wedge \omega_{q'}.$$

Now we use a known theorem (see, for example, Stewart 1971)

$$\int_{P(x)} p^{\alpha_n} \mathscr{L}(\boldsymbol{X}) f(x, p) \omega_p = \nabla_\mu \int_{P(x)} p^{\alpha_n} p^\mu f(x, p) \omega_p$$

and also substitute the truncated development (7) in (16), for $f(x, p)$:

$$f \approx f_0 \sum_{n=0}^{2} a_{\alpha_n} H^{\alpha_n}$$

thus getting

$$\int p^{\alpha_n} \mathscr{L}(\boldsymbol{X}) f(x, p) \omega_p = \sum_{r,s} a_{\beta_r} a_{\gamma_s} (C^{\alpha_n \beta_r \gamma_s}(x) - D^{\alpha_n \beta_r \gamma_s}(x)) \tag{17}$$

where

$$C^{\alpha_n \beta_r \gamma_s}(x) \equiv \int p^{\alpha_n} H^{\beta_r}(p') H^{\gamma_s}(q') f_0(p') f_0(q') W_{\mathrm{I}} \omega^4$$

$$D^{\alpha_n \beta_r \gamma_s}(x) \equiv \int p^{\alpha_n} H^{\beta_r}(p) H^{\gamma_s}(q) f_0(p) f_0(q) W_{\mathrm{D}} \omega^4.$$

From (17) one gets, independently of assumption (3),

$$\nabla_\mu N^\mu = 0$$

and when (3*b*) holds

$$\nabla_\mu T^{\alpha\mu} = 0. \tag{17a}$$

If we now linearize, i.e. eliminate the products of type $a_{\beta_r} a_{\gamma_s}$ with r or $s \neq 0$, from the calculation, we finally get

$$\nabla_\mu \int p^{\alpha_n} p^\mu f(x, p) \omega_p = \sum_{r=0}^{2} a_{\beta_r} B^{\alpha_n \beta_r} \tag{18}$$

where

$$B^{\alpha_n \beta_r} \equiv C^{\alpha_n \beta_r 0} + C^{\alpha_n 0 \beta_r} - D^{\alpha_n \beta_r 0} - D^{\alpha_n 0 \beta_r}. \tag{18b}$$

One can show, independently of assumption (3), that

$$B^{0,0} = B^{0,\mu} = B^{0,\mu\nu} = g_{\alpha\beta} B^{\alpha\beta,\mu} = g_{\alpha\beta} B^{\alpha\beta,\mu\nu} = 0. \tag{18c}$$

and from the self-orthogonality of the Grad polynomials, $g_{\mu\nu} H^{\mu\nu} = 0$

$$g_{\mu\nu} B^{\alpha,\mu\nu} = g_{\mu\nu} B^{\alpha\beta,\mu\nu} = 0. \tag{18d}$$

All the information we get from the Boltzmann equation is then contained in equation (18). In order to later obtain a more manageable set of equations, we decompose the first 'collision integrals' $B^{\alpha_n\beta_r}$ in terms of the basic tensors at our disposal, $g^{\alpha\beta}$ and u^α, in the known form:

$$\begin{aligned}
B^{\alpha,\mu} &\equiv {}^{11}B_1 u^\alpha u^\mu + {}^{11}B_2 g^{\alpha\mu} \\
B^{\alpha,\mu\nu} &\equiv {}^{12}B_1 u^\alpha u^\mu u^\nu + (\tfrac{1}{4}\,{}^{12}B_1 - \tfrac{1}{2}\,{}^{12}B_2) u^\alpha g^{\mu\nu} + 2\,{}^{12}B_2 g^{\alpha(\mu} u^{\nu)} \\
B^{\alpha\beta,\mu} &\equiv {}^{21}B_1 u^\alpha u^\beta u^\mu + (\tfrac{1}{4}\,{}^{21}B_1 - \tfrac{1}{2}\,{}^{21}B_2) g^{\alpha\beta} u^\mu + 2\,{}^{21}B_2 g^{\mu(\alpha} u^{\beta)} \\
B^{\alpha\beta,\mu\nu} &\equiv {}^{22}B_1 u^\alpha u^\beta u^\mu u^\nu + {}^{22}B_2 g^{\alpha\beta} u^\mu u^\nu + {}^{22}B_3 u^\alpha u^\beta g^{\mu\nu} + {}^{22}B_4 u^{(\alpha} g^{\beta)(\mu} u^{\nu)} \\
&\qquad + {}^{22}B_5 g^{\alpha(\mu} g^{\nu)\beta} + {}^{22}B_6 g^{\alpha\beta} g^{\mu\nu}.
\end{aligned} \tag{19}$$

We shall also make use of the third moment of the zeroth-order distribution function, which we decompose in a similar manner:

$$\begin{aligned}
S_0^{\alpha\beta\mu} &\equiv \int f_0 p^\alpha p^\beta p^\mu \omega_p \equiv \omega_0 u^\alpha u^\beta u^\mu + 3\xi_0 u^{(\alpha} g^{\beta\mu)} \\
S^{\alpha\beta\mu} &\equiv \int f p^\alpha p^\beta p^\mu \omega_p \equiv \omega u^\alpha u^\beta u^\mu + 3\xi u^{(\alpha} g^{\beta\mu)}.
\end{aligned} \tag{20}$$

For $n = 0, 1, 2$, equations (18) then clearly read

$$\begin{aligned}
\nabla_\mu N^\mu &= 0 \\
\nabla_\mu T^{\alpha\mu} &= B^{\alpha,0} + a_\mu B^{\alpha,\mu} + a_{\mu\nu} B^{\alpha,\mu\nu} \\
\nabla_\mu S^{\alpha\beta\mu} &= B^{\alpha\beta,0} + a_\mu B^{\alpha\beta,\mu} + a_{\mu\nu} B^{\alpha\beta,\mu\nu}.
\end{aligned} \tag{21}$$

We introduce the notation

$$\begin{aligned}
\nabla_\mu T^{\alpha\mu} &\equiv B^{\alpha,0} + \nabla_\mu \hat{T}^{\alpha\mu} \\
\nabla_\mu S^{\alpha\beta\mu} &\equiv B^{\alpha\beta,0} + \nabla_\mu \hat{S}^{\alpha\beta\mu}
\end{aligned}$$

so that we can write equations (21) in the form

$$\begin{aligned}
\nabla_\mu N^\mu &= 0 \\
\nabla_\mu \hat{T}^{\alpha\mu} &= a_\mu B^{\alpha,\mu} + a_{\mu\nu} B^{\alpha,\mu\nu} \\
\nabla_\mu \hat{S}^{\alpha\beta\mu} &= a_\mu B^{\alpha\beta,\mu} + a_{\mu\nu} B^{\alpha\beta,\mu\nu}
\end{aligned} \tag{22}$$

which we call 'generalized Grad equations'.

When 'PT invariance' (i.e. equations ($3a$), ($3b$)) holds, some collision integrals $B^{\alpha_n\beta_r}$ reduce to zero. Explicitly, when ($3b$) holds

$$B^{\alpha,\beta_r} = 0 \tag{23}$$

and when ($3a$) holds and, in addition, $Q(f_0, f_0) = 0$

$$B^{\alpha\beta,0} = 0,$$

so that

$$\nabla_\mu T^{\alpha,\mu} = \nabla_\mu \hat{T}^{\alpha\mu}; \qquad \nabla_\mu S^{\alpha\beta\mu} = \nabla_\mu \hat{S}^{\alpha\beta\mu}.$$

If, in addition, we suppose the 'Eckart matching condition', $N^\alpha = N_0^\alpha$, that we have seen to imply (15*b*) $a_\mu = 0$, we recover the usual relativistic Grad equations, as given by Anderson and Stewart:

$$\begin{aligned} &\nabla_\mu N^\mu = 0 \\ &\nabla_\mu T^{\alpha\mu} = 0 \\ &\nabla_\mu S^{\alpha\beta\mu} = a_{\mu\nu} B^{\alpha\beta,\mu\nu} \end{aligned} \tag{24}$$

However, our aim is to solve the more general equations (22). In order to do this, we make use of an approximation which, in our opinion, is the most natural generalization of the usual normal solution approximation.

3. The generalized normal solution approximation

Let us recall (Stewart 1971) the way in which the normal solution approximation is introduced in the usual ('T-invariant') situation (i.e., when equations (24) are valid). The fourth equation of the set (10) (i.e., the one obtained for $n = 3$) is

$$M^{\alpha\beta\gamma\delta\epsilon\mu} a_{\delta\epsilon\mu} = S^{\alpha\beta\gamma} - S_0^{\alpha\beta\gamma} - \alpha_1{}^{\alpha\beta\gamma}{}_{\delta\epsilon}(T^{\delta\epsilon} - T_0^{\delta\epsilon})$$

where $\alpha_1{}^{\alpha\beta\gamma}{}_{\delta\epsilon}$ is defined in (9*b*). But we have truncated our series (7) at $n = 3$, so $a_{\delta\epsilon\mu} \equiv 0$. This implies

$$S^{\alpha\beta\gamma} = S_0^{\alpha\beta\gamma} + \alpha_1{}^{\alpha\beta\gamma}{}_{\delta\epsilon}(T^{\delta\epsilon} - T_0^{\delta\epsilon})$$

and the third equation of the set (24) reduces to

$$\nabla_\gamma S_0^{\alpha\beta\gamma} + \nabla_\gamma[\alpha_1{}^{\alpha\beta\gamma}{}_{\delta\epsilon}(T^{\delta\epsilon} - T_0^{\delta\epsilon})] = a_{\mu\nu} B^{\alpha\beta,\mu\nu} \tag{24b}$$

which is to be solved for $\Delta T^{\delta\epsilon} \equiv T^{\delta\epsilon} - T_0^{\delta\epsilon}$ in conjunction with $M^{\alpha\beta\gamma\delta} a_{\gamma\delta} = \Delta T^{\alpha\beta}$ (obtained from the third equation of (11) when the Eckart matching conditions are introduced). When the second term of the left-hand side of (24*b*) is neglected, this equation can be solved algebraically for $\Delta T^{\delta\epsilon}$ in terms of gradients of the equilibrium quantities, i.e. we obtain a normal solution.

We can then say that the usual 'normal solution approximation' formally consists of writing the third equation of (24) as

$$\nabla_\mu S_0^{\alpha\beta\mu} = a_{\mu\nu} B^{\alpha\beta,\mu\nu} \tag{25}$$

i.e., we suppose that the covariant derivatives of the third moments of the two distribution functions are equal (to this order of approximation):

$$\nabla_\mu S^{\alpha\beta\mu} = \nabla_\mu S_0^{\alpha\beta\mu}.$$

However, in the left-hand side of our equations (22) $\nabla_\mu \hat{S}^{\alpha\beta\mu}$ appears instead of $\nabla_\mu S^{\alpha\beta\mu}$. The natural approximation in our situation is, then,

$$\nabla_\mu \hat{S}^{\alpha\beta\mu} = \nabla_\mu S_0^{\alpha\beta\mu}. \tag{26}$$

This approximation is valid only when both the T violation is 'weak' (i.e. we can write $\nabla_\mu \hat{S}^{\alpha\beta\mu} \sim \nabla_\mu S^{\alpha\beta\mu}$, *and* we can also write (25) (i.e. the microscopic time and length scales are much shorter than the macroscopic ones). We shall call this approximation (26), and the equations obtained from it, the 'first level' of the approximation, because we can introduce another level, i.e. in addition to (26), the local 'conservation equations'

$$\nabla_\mu \hat{T}^{\alpha\mu} = 0. \tag{27}$$

We shall call calculations done with (26) and (27) the 'second level' approximation†.

Introducing now the decompositions (15), (19) and

$$\begin{aligned} &\nabla_\mu \hat{T}^{\alpha\mu} \equiv \hat{T}_1 u^\alpha + \hat{T}_2^\alpha; \qquad \hat{T}_2^\alpha u_\alpha = 0 \\ &\nabla_\mu S_0{}^{\alpha\beta\mu} \equiv S_0 u^\alpha u^\beta + S_1 g^{\alpha\beta} + 2S_2{}^{(\alpha} u^{\beta)} + S_3^{\alpha\beta} \\ &S_2{}^\alpha u_\alpha = S_3{}^\alpha{}_\alpha = S_3{}^{\alpha\beta} u_\beta = 0 \end{aligned} \tag{28}$$

in our generalized Grad equations (22), we get

$$\begin{aligned} &4\hat{T}_1 = 3L_0({}^{12}B_1 - 2\,{}^{12}B_2) \\ &\hat{T}_2^\alpha = K_2^\alpha\,{}^{11}B_2 - 2L_2^\alpha\,{}^{12}B_2 \\ &4S_0 = L_0[3({}^{22}B_1 - {}^{22}B_4) + 4\,{}^{22}B_5] \\ &4S_1 = L_0(3\,{}^{22}B_2 + {}^{22}B_5) \\ &2S_2^\alpha = 2\,{}^{21}B_2 K_2^\alpha + (2\,{}^{22}B_5 - {}^{22}B_4)L_2^\alpha \\ &S_3^{\alpha\beta} = {}^{22}B_5 L_3^{\alpha\beta}. \end{aligned} \tag{29}$$

It is obvious that

$$g_{\alpha\beta} \nabla_\rho S_0^{\alpha\beta\rho} = -m^2 \nabla_\rho N_0^\rho = 0$$

so that the coefficients of the decomposition (28) are not at all independent; they must verify $S_0 = 4S_1$. As these coefficients are related to the 'collision integrals' ${}^{ab}B_c$ by the generalized Grad equations (29) there is an integrability condition, i.e.

$${}^{22}B_1 - {}^{22}B_4 = 4\,{}^{22}B_2.$$

This condition is identically verified, because the equations (18*c*) and (18*d*) impose on the coefficients of the decomposition (19) of $B^{\alpha\beta,\mu\nu}$ the conditions

$${}^{22}B_3 = {}^{22}B_2 \qquad -{}^{22}B_1 + 4\,{}^{22}B_2 + {}^{22}B_4 = 0 \qquad -{}^{22}B_3 + {}^{22}B_5 + 4\,{}^{22}B_6 = 0.$$

We note that the 'second level' approximation imposes some restrictions on the coefficients ${}^{ab}B_c$: (27) implies $\hat{T}_1 = 0$, and from the first equation of (29) we get ${}^{12}B_1 = 2\,{}^{12}B_2$, another integrability condition, which can be interpreted as a condition on the form of the admissible scattering matrices. A sufficient condition for the verification of this equation would be the verification of T invariance.

4. The transport equations

We begin by analyzing the first equations of the set (29). One can easily show (using (14)) that when restoring PT invariance (we shall denote this by lim(PT)), this equation

† The physical interpretation of the approximations made must possibly be a kind of series development in two parameters: one related to the quotient between microscopic and macroscopic time scales and the other related to the non-conserved 'momentum' $t^\alpha \equiv p^\alpha + q^\alpha - p'^\alpha - q'^\alpha$. Of course, in the usual 'T-invariant' situation, $t^\alpha \equiv 0$, and only one 'parameter' is relevant: both levels of our approximation reduce to formula (25).

reduces to the usual expression for the local conservation of energy:

$$\dot{\mu}+(\mu+p)\theta+\pi^{\alpha\beta}\sigma_{\alpha\beta}+q^{\alpha}_{;\alpha}+q^{\alpha}\dot{u}_{\alpha}=0 \tag{30}$$

where $\theta(\equiv u^{\alpha}_{;\alpha})$ is the scalar expansion rate, $\omega_{\alpha\beta}(\equiv u_{[\alpha;\beta]}+\dot{u}_{[\alpha}u_{\beta]})$ is the vorticity tensor, $\sigma_{\alpha\beta}(\equiv u_{(\alpha;\beta)}+\dot{u}_{(\alpha}u_{\beta)}-\frac{1}{3}\theta h_{\alpha\beta})$ is the shear tensor, $\omega(\equiv(\frac{1}{2}\omega_{\mu\nu}\omega^{\mu\nu})^{1/2})$ is the vorticity scalar, and $\sigma(\equiv(\frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu})^{1/2})$ is the shear scalar.

In the general situation (where neither (3a) nor (3b) holds), this equation is then a kind of 'diffusion equation' and can be written as

$$\dot{\mu}+(\mu+p)\theta+\pi^{\alpha\beta}\sigma_{\alpha\beta}+q^{\alpha}_{;\alpha}+q^{\alpha}\dot{u}_{\alpha}=\tfrac{3}{4}L_0({}^{12}B_1-2\,{}^{12}B_2). \tag{31}$$

Next, we solve the other equations of the set (29). To do this, we recall that for the Jüttner–Synge function the coefficients of the decomposition (28) can be written as

$$\begin{aligned}
&\tfrac{1}{4}S_0=S_1=\xi_0\theta(\tfrac{5}{3}-\Gamma-\Gamma/\gamma h)\\
&S_2^{\alpha}=-\lambda h^{\alpha\rho}\left[\frac{\dot{u}_\rho}{\gamma}+\left(\frac{1}{\gamma}\right)_{;\rho}\right]\\
&\lambda\equiv mp_0\frac{\mathrm{d}h}{\mathrm{d}\gamma}\gamma^2\\
&S_3^{\alpha\beta}=2\xi_0\sigma^{\alpha\beta}
\end{aligned} \tag{32}$$

where $\gamma\equiv m/kT$ and the relativistic enthalpy, $h(\gamma)$, is defined by

$$h(\gamma)\equiv\frac{\mu_0+p_0}{mp_0}\equiv\frac{K_3(\gamma)}{K_2(\gamma)},$$

$K_n(r)$ being the Kelvin function of order n; Γ is the ratio of specific heats

$$\Gamma\equiv\frac{c_p}{c_v}=\frac{\gamma^2h'}{\gamma^2h'+1};\qquad h'\equiv\frac{\mathrm{d}h}{\mathrm{d}\gamma}.$$

A comparison of (32), (29) and (15) immediately yields

$$\begin{aligned}
&\pi^{\alpha\beta}=-\eta\sigma^{\alpha\beta}\\
&\eta\equiv-2\xi_0(M_5/{}^{22}B_5)
\end{aligned} \tag{33}$$

and

$$\tau=\phi\theta \tag{34}$$

with

$$\phi\equiv\frac{3M_2+M_5}{3\,{}^{21}B_2+{}^{22}B_5}\xi_0(\tfrac{5}{3}-\Gamma-\Gamma/\gamma h).$$

These two equations, (33) and (34), are exactly the same as the usual ones. One can then say that the shear and bulk viscosities are not affected by the removal of the T invariance hypothesis (3).

Using the decompositions (14) for $T^{\alpha\beta}$ and $T_0^{\alpha\beta}$ it is not difficult to show that

$$\hat{T}_{2\alpha}=h_{\alpha\beta}(T^{\rho\beta}-T_0^{\rho\beta})_{;\rho}=l_\alpha+t_\alpha$$

with

$$\begin{aligned}
&l_\alpha\equiv q_\alpha+q^{\rho}(\tfrac{4}{3}\theta h_{\rho\alpha}-u_\alpha\dot{u}_\rho+w_{\alpha\rho}+\sigma_{\alpha\rho})-2\pi_{\alpha;\mu}^{\mu}+4\eta\sigma^2u_\alpha\\
&t_\alpha\equiv\partial_\alpha\tau+\dot{\tau}u_\alpha+4\tau\dot{u}_\alpha.
\end{aligned}$$

Next, by using the following equations of the set (29):

$$\hat{T}_2^\alpha = K_2^\alpha\, {}^{11}B_2 - 2\, {}^{12}B_2 L_2^\alpha$$

$$2S_2^\alpha = 2\, {}^{21}B_2 K_2^\alpha + (2\, {}^{22}B_5 - {}^{22}B_4) L_2^\alpha$$

and the fifth of (15),

$$2q^\alpha = (2M_5 - M_4) L_2^\alpha + c_3 M_2 K_2^\alpha$$

one gets the 'first level' heat equation

$$\begin{aligned} q_\rho[(1+\tfrac{4}{3}D_6D_3\theta)h^{\rho\alpha} + D_6D_3(\omega^{\alpha\rho} + \sigma^{\alpha\rho} - u^\alpha \dot{u}^\rho)] + D_6D_3\dot{q}^\alpha \\ = 2D_6D_3(\eta\sigma^{\alpha\mu})_{;\mu} - 4\eta D_6D_3\sigma^2 u^\alpha - D_6D_3 h^{\alpha\rho}\tau_{;\rho} \\ - 4D_6D_3\tau\dot{u}^\alpha - \lambda D_3 h^{\alpha\rho}\left[\frac{\dot{u}_\rho}{\gamma} + \left(\frac{1}{\gamma}\right)_{;\rho}\right] \end{aligned} \tag{35}$$

where

$$D_6D_3 \equiv \frac{2\, {}^{21}B_2(2M_5 - M_4) - c_3M_2({}^{22}B_5 - {}^{22}B_4)}{2[4\, {}^{12}B_2\, {}^{21}B_2 + {}^{11}B_2(2\, {}^{22}B_5 - {}^{22}B_4)]}$$

$$D_3 \equiv \frac{{}^{11}B_2(2M_5 - M_4) - c_3M_2({}^{22}B_5 - {}^{22}B_4)}{4\, {}^{12}B_2\, {}^{21}B_2 + {}^{11}B_2(2\, {}^{22}B_5 - {}^{22}B_4)}.$$

When there is no energy flux different from the heat flux (this situation leads classically to the Fourier equation, so we shall call it the 'Fourier situation'), i.e. $\sigma^{\alpha\beta} = \omega^{\alpha\beta} = \theta = \dot{u}^\alpha = 0$, our equation (35) reduces to

$$q^\alpha + D_6D_3\dot{q}^\alpha = -\lambda D_3 h^{\alpha\rho}\partial_\rho(1/\gamma). \tag{36}$$

This equation (36) is formally very similar to that proposed by Cattaneo (1958) and Vernotte (1958) for solving the problem of the parabolicity of the heat equation. In fact, the equation for the propagation of temperature associated with (36) is hyperbolic whenever $\kappa \equiv D_6D_3$, the 'second heat conduction coefficient', is positive†.

In the second level of the generalized normal solution approximation

$$\hat{T}_2^\alpha = 0$$

which implies, from the third equation of the set (29)

$$K_2^\alpha = 2rL_2^\alpha \qquad r \equiv {}^{12}B_2({}^{11}B_2)^{-1}.$$

Substituting it in

$$2S_2^\alpha = 2\, {}^{21}B_2 K_2^\alpha + (2\, {}^{22}B_5 - {}^{22}B_4) L_2^\alpha$$

and

$$2q^\alpha = (2M_5 - M_4) L_2^\alpha + c_3 M_2 K_2^\alpha$$

† In fact is it not clear that the problem of the infinite velocity for the heat propagation is of relativistic origin; it can equally well be interpreted as a result of the way in which transport equations are obtained (expansion in powers of a small parameter) implying that physical phenomena are dealt with on a new scale of length and/or time. From this point of view the basic reason why the velocity of heat propagation is infinite is that these changes of scale do not preserve the light cone but rather tend to make it flat. I am grateful to the referee for clarifying this point for me.

one gets the 'second level heat equation'

$$q^{\alpha} = -\bar{\lambda} h^{\alpha\rho}\left[\frac{\dot{u}_{\rho}}{\gamma} + \left(\frac{1}{\gamma}\right)_{;\rho}\right] \tag{37}$$

where

$$\bar{\lambda} \equiv mp_0 h' \gamma^2 \frac{2M_5 - M_4 + 2rc_3 M_{12}}{2\,{}^{22}B_5 - {}^{22}B_4 + 4r\,{}^{21}B_2}.$$

Equation (37) is identical to the usual Eckart-like one, but the expressions for $\bar{\lambda}$ coincide only when $r = 0$†.

5. A Bel-like gas

We can also carry out these calculations for a Bel-like gas, i.e. one for which the distribution function is a 'perturbation' of the Bel function (Bel 1969, Alvarez 1976)

$$f_0 = B(x)\exp(\mu_{\alpha\beta} p^{\alpha} p^{\beta}) \tag{38}$$

with

$$\mu_{\alpha\beta} p^{\alpha} p^{\beta} = -z^2 m^{-2}(E^2 - m^2).$$

It is easily shown that a Bel-like gas satisfies

$$\begin{aligned}
n_0 &= \pi\sqrt{\pi} B m^3 z^{-3/2} \\
\mu_0 &= \pi\sqrt{\pi} B m^4 U(\tfrac{3}{2}, 3, z) \\
3p_0 &= \pi\sqrt{\pi} B m^4 (U(\tfrac{3}{2}, 3, z) - U(\tfrac{3}{2}, 2, z)) \\
\xi_0 &= \tfrac{1}{2} n_0 m^2 z^{-1} \\
\omega_0 &= m^2 n_0 + 6\xi_0
\end{aligned} \tag{39}$$

where $U(a, b, z)$ is the Kummer function, and the physical interpretation of the variable z in uniform space–times

$$ds^2 = -(dx^0)^2 + R^2 K^2 \delta_{ij}\, dx^i\, dx^j$$

is $z = \zeta^2 R^2 m^2$, ζ being a constant, related to the inverse of the 'temperature' by the formula $\zeta^2 R^2 = (2mkT)^{-1}$. In order to obtain transport equations, we shall assume in this paper that f_0 is only a 'local' equilibrium function, i.e. both z and B are functions of the position.

† We note that when one uses the 'Eckart' matching conditions $N^{\alpha} = N_0^{\alpha}$ (i.e. $a_{\alpha} = 0$, which implies $K_1 = K_{\alpha} = 0$) our generalized Grad equations (29) are soluble only when PT invariance is restored, i.e. when these equations read

$$\begin{aligned}
\hat{T}_2^{\alpha} &= 0 \\
2S_2^{\alpha} &= (2\,{}^{22}B_5 - {}^{22}B_4) L_2^{\alpha}
\end{aligned}$$

which imply equation (37) with

$$\bar{\lambda} = \frac{2M_{25} - M_{24}}{2\,{}^{22}B_5 - {}^{22}B_4}.$$

In order for the first two moments N_0^α and $T_0^{\alpha\beta}$ to be conserved, we must have

$$\begin{aligned} &\frac{3\dot{y}}{y}-\frac{\dot{B}}{B}=\theta \\ &\dot{B}=0 \\ &p_0 h^\mu{}_\nu \frac{\partial^\nu B}{B}=-(\mu_0+p_0)\dot{u}^\mu+h^\mu{}_\nu(\mu_0-(2m^2y^2-1)p_0)\frac{\partial^\nu y}{y} \end{aligned} \tag{40}$$

where $y\equiv m^{-1}z^{1/2}$ and $\dot{B}\equiv u^\mu\nabla_\mu B$.

We can also easily show that for an f_0 given by (38) the coefficients defined in (28) are given by

$$\begin{aligned} &S_0=S_1=0 \\ &S_3^{\alpha\beta}=2\xi_0\sigma^{\alpha\beta} \\ &S_2^\alpha=Vh^\alpha{}_\rho\left(\dot{u}^\rho-\frac{\partial^\rho y}{y}\right) \end{aligned} \tag{41}$$

where

$$V\equiv\xi_0(2m^2y^2+4-\mu_0p_0^{-1}).$$

Now† we can solve the generalized grad equations (29) for our Bel gas, getting

$$\tau=0$$

(which is perhaps an interesting result, since it implies that the bulk viscosity is null for a Bel-like gas) and equations (33), (35) and (37) which we have obtained for a Jüttner–Synge-like gas; the only difference being that the variable γ is substituted by y and

$$\lambda_{(\mathrm{Bel})}=y\xi_0(\mu_0p_0^{-1}-2m^2y^2-4) \tag{42}$$

and

$$\bar{\lambda}_{(\mathrm{Bel})}=\lambda_{(\mathrm{Bel})}D_3. \tag{43}$$

Thus, the usual coefficient of thermal conductivity, $\bar{\lambda}$, becomes null not only when $D_3=0$, but also if

$$\mu_0=2(m^2y^2+2)p_0=[(m/kT)+4]p_0.$$

Of course, a glance at equations (39) shows that this 'equation of state' (as the one for which $\bar{\lambda}_{\mathrm{JS}}$ is null, i.e. $h(\gamma)=$ constant) is never exactly satisfied for a Bel gas.

6. Conclusions

In order to calculate the limit of our first level heat equation (35) when PT invariance is restored, it is useful to write it in the form

$$q^\alpha+D_6D_3\hat{T}_2^\alpha=D_3S_2^\alpha. \tag{44}$$

† It is to be noted that the use of the generalized normal solution approximation has, in the Bel situation, a different meaning than in the Jüttner–Synge one, owing to the fact that $Q(f_{\mathrm{Bel}},f_{\mathrm{Bel}})\neq 0$.

We first note that

$$\lim(\mathrm{PT})D_3 = \frac{2M_5 - M_4 + 2c_3\hat{r}M_2}{4\hat{r}\,{}^{21}B_2 + 2\,{}^{22}B_5 - {}^{22}B_4}$$

and

$$\lim(\mathrm{PT})(D_6 D_3 \hat{T}_2^\alpha) = \frac{2\,{}^{21}B_2(2M_5 - M_4) - c_3 M_2({}^{22}B_5 - {}^{22}B_4)}{2(4\hat{r}\,{}^{21}B_2 + 2\,{}^{22}B_5 - {}^{22}B_4)}(K_2^\alpha - 2\hat{r}L_2^\alpha)$$

denoting by $\hat{r} \equiv \lim(\mathrm{PT})r$. Thus we see that we recover the usual Eckart equation only when $\hat{r} = 0$ and $K_2^\alpha = 0$. This result could easily be expected, because when the PT symmetry for the scattering matrix holds, the two levels of our generalized normal solution approximation are equivalent; the lim(PT) of equation (37)then being the same as that of equation (35).

To summarize our results, we have obtained a new set of Grad equations (29), generalizing the usual ones when PT invariance does not hold. To solve these equations, we have introduced the normal solution approximation in two levels; in both levels we recover the usual formulae for the shear and bulk viscosities. In the first level, we have obtained a new heat equation that leads to a hyperbolic heat equation in a 'Fourier situation', and in the second level it reduces to the usual Eckart-like equation when PT invariance is restored.

We have also obtained the result that the bulk viscosity of a Bel-like gas is zero, which agrees with the fact that this function is compatible with non-stationary space–times. The physical meaning of this result is that a cosmological expansion described by the Bel function is isentropic (in the usual sense of vector entropy; but in the alternative description in terms of a scalar 'proper time entropy' (Alvarez 1976) this would not be the case). The existence of this kind of 'isentropic' cosmological expansion was claimed by Schücking and Spiegel (1970). We will discuss this kind of problem in a forthcoming paper.

However, we have not solved the problem of the 'parabolicity' of the relativistic heat equation in the usual situation in which PT invariance holds, and, furthermore, in view of our results, it seems impossible to make PT invariance compatible with the non-conservation of the energy–momentum tensor in the framework of this simple theory.

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